

MAS212 Linear Algebra I Exercises 5 (with solutions)

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Assessed submission deadline 5pm on Friday 3rd November 2006 in the blue box on the ground floor.

Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together.

Marks out of 100 are shown in square brackets after each question number.

1. [20] Prove that each of the following maps either is or is not linear:

- (a) $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}, \alpha(x, y, z) = x - y + z$;
- (b) $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \alpha(x, y, z) = (x, x + y, x + y + z)$;
- (c) $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}, \alpha(x, y) = x + y + 1$;
- (d) $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}, \alpha(x_1, \dots, x_n) = x_1 x_2 x_3 \cdots x_n$ for integer $n > 1$.

Solution

(a) is *linear* because $\forall (x, y, z), (x', y', z') \in \mathbb{R}^3$,

$$\alpha((x, y, z) + (x', y', z')) = \alpha(x + x', y + y', z + z') = x + x' - y - y' + z + z' = (x - y + z) + (x' - y' + z') = \alpha(x, y, z) + \alpha(x', y', z')$$

and $\forall (x, y, z) \in \mathbb{R}^3, k \in \mathbb{R}$,

$$\alpha(k(x, y, z)) = \alpha(kx, ky, kz) = kx - ky + kz = k(x - y + z) = k\alpha(x, y, z).$$

(b) is *linear* because $\forall (x, y, z), (x', y', z') \in \mathbb{R}^3$,

$$\begin{aligned} \alpha((x, y, z) + (x', y', z')) &= \alpha(x + x', y + y', z + z') = (x + x', x + x' + y + y', x + x' + y + y' + z + z') \\ &= (x, x + y, x + y + z) + (x', x' + y', x' + y' + z') = \alpha(x, y, z) + \alpha(x', y', z') \end{aligned}$$

and $\forall (x, y, z) \in \mathbb{R}^3, k \in \mathbb{R}$,

$$\begin{aligned} \alpha(k(x, y, z)) &= \alpha(kx, ky, kz) = (kx, kx + ky, kx + ky + kz) = k(x, x + y, x + y + z) \\ &= k\alpha(x, y, z). \end{aligned}$$

(c) is *nonlinear* because $\alpha(0, 0) = 1 \neq 0$.

(d) is *nonlinear* because

$$\begin{aligned} \forall (x_1, \dots, x_n) \in \mathbb{R}^n, k \in \mathbb{R}, \alpha(k(x_1, \dots, x_n)) &= \alpha(kx_1, \dots, kx_n) = kx_1 kx_2 kx_3 \cdots kx_n = \\ &= k^n \alpha(x_1, \dots, x_n) \neq k\alpha(x_1, \dots, x_n) \text{ if } n > 1. \text{ (Or choose a more specific counterexample.)} \end{aligned}$$

2. [40] Find basis sets for the kernel and image of each of the following linear maps; hence write down the nullity and rank of each map and show that the dimension theorem for linear maps is satisfied. [You are not required to prove any claims about basis sets.]

- (a) $\alpha: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\alpha(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 0)$;
 (b) $\alpha: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\alpha(x_1, x_2) = (x_1, x_1 - x_2)$;
 (c) $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$, $\alpha(x_1, \dots, x_n) = x_1 + x_n$;
 (d) $\alpha: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\alpha(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 - 3x_2 + 4x_3, x_1 + 18x_2 - 17x_3)$.

Solution

(a) $\ker(\alpha)$ is defined by $\alpha(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 0) = (0, 0) \Rightarrow x_3 = -x_1 + x_2$,

hence $\ker(\alpha) = \{(x_1, x_2, -x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\} = \{x_1(1, 0, -1) + x_2(0, 1, 1) \mid x_1, x_2 \in \mathbb{R}\} = \langle(1, 0, -1), (0, 1, 1)\rangle$ and $\{(1, 0, -1), (0, 1, 1)\}$ is clearly linearly independent and so is a basis.
 $\text{im}(\alpha) = \{(x_1 - x_2 + x_3, 0) \mid x_1, x_2, x_3 \in \mathbb{R}\} = \{(x, 0) \mid x \in \mathbb{R}\} = \langle(1, 0)\rangle$ so $\{(1, 0)\}$ is clearly a basis.

Hence *nullity* = 2 and *rank* = 1. The domain is $\mathbb{R}^3$, so $\dim(\text{domain}) = 3 = 2 + 1$ and the dimension theorem is satisfied.

(b) $\ker(\alpha)$ is defined by $\alpha(x_1, x_2) = (x_1, x_1 - x_2) = (0, 0)$, hence $\ker(\alpha) = \{(0, 0)\} = \langle(0, 0)\rangle$ but the basis is the empty set $\{\}$ since there are no linearly independent vectors in this (trivial) zero vector subspace.

$\text{im}(\alpha) = \{(x_1, x_1 - x_2) \mid x_1, x_2 \in \mathbb{R}\} = \{x_1(1, 1) + x_2(0, -1) \mid x_1, x_2 \in \mathbb{R}\} = \langle(1, 1), (0, -1)\rangle$ and $\{(1, 1), (0, -1)\}$ is clearly linearly independent and so is a basis.

Hence *nullity* = 0 and *rank* = 2. The domain is $\mathbb{R}^2$, so $\dim(\text{domain}) = 2 = 0 + 2$ and the dimension theorem is satisfied.

(c) $\ker(\alpha)$ is defined by $\alpha(x_1, \dots, x_n) = x_1 + x_n = 0$, hence $\ker(\alpha) = \{(x_1, x_2, \dots, x_{n-1}, -x_1) \mid x_1, \dots, x_{n-1} \in \mathbb{R}\} = \langle(1, 0, 0, \dots, -1), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1, 0)\rangle$ so $\{(1, 0, 0, \dots, -1), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1, 0)\}$ is clearly a basis.

The image is $\{(x_1 + x_n) \mid x_1, x_n \in \mathbb{R}\} = \{(x) \mid x \in \mathbb{R}\} = \langle(1)\rangle$ and $\{(1)\}$ is clearly a basis for $\text{im}(\alpha) = \mathbb{R}$. [The vector $(x) \in \mathbb{R}^1$ may also be written as simply $x \in \mathbb{R}$.]

Hence *nullity* = $n - 1$ and *rank* = 1. The domain is $\mathbb{R}^n$, so $\dim(\text{domain}) = n = (n - 1) + 1$ and the dimension theorem is satisfied.

(d) $\ker(\alpha)$ is defined by $\alpha(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 - 3x_2 + 4x_3, x_1 + 18x_2 - 17x_3) = (0, 0, 0)$, which (by solving the equations) gives $\ker(\alpha) = \{(x, -x, -x) \mid x \in \mathbb{R}\} = \langle(1, -1, -1)\rangle$.

$\text{im}(\alpha) = \{(2x_1 + x_2 + x_3, x_1 - 3x_2 + 4x_3, x_1 + 18x_2 - 17x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\} = \{x_1(2, 1, 1) + x_2(1, -3, 18) + x_3(1, 4, -17) \mid x_1, x_2, x_3 \in \mathbb{R}\} = \langle(2, 1, 1), (1, -3, 18), (1, 4, -17)\rangle$.

vectors are clearly not linearly independent, since the first minus the second gives the third, so take the first two as a basis for the image, namely $\{(2, 1, 1), (1, -3, 18)\}$.

Hence *nullity* = 1 and *rank* = 2. The domain is $\mathbb{R}^3$, so $\dim(\text{domain}) = 3 = 1 + 2$ and the dimension theorem is satisfied.

3. [20] (a) Prove that the set $\mathbb{C}$ of complex numbers of the form $x + iy$, where $x, y \in \mathbb{R}$, with the usual rules for complex number addition and multiplication, is a vector space over the field of real numbers $\mathbb{R}$. [You may use the fact that $\mathbb{C}$ is a field to shorten the proof considerably.] Write down a basis for this vector space and hence state its dimension.

(b) Let $M_{2,2}(\mathbb{R})$ be the vector space of 2×2 matrices over $\mathbb{R}$, with the usual rules for matrix addition and scalar multiplication, and let $\mathbb{C}$ be the vector space of complex numbers over $\mathbb{R}$. Show that the

map $\alpha : \mathbb{C} \rightarrow M_{2,2}(\mathbb{R})$ defined by $\alpha(x + iy) = \begin{bmatrix} x & y \\ -y & -x \end{bmatrix}$ is linear.

Solution

(a) The complex numbers are a field so they satisfy the additive axioms for a vector space (which are the axioms of an additive group). A real number is a special case of a complex number and the complex numbers are a field, so they satisfy the scalar multiplication axioms for a vector space over the real field. (In fact, any field is trivially a vector space over itself.) The obvious basis is $\{1 + i0, 0 + i1\}$, which is usually written more succinctly as $\{1, i\}$, so the dimension of $\mathbb{C}$ as a vector space over $\mathbb{R}$ is 2.

$$(b) \quad \forall (x_1 + iy_1), (x_2 + iy_2) \in \mathbb{C}, \alpha((x_1 + iy_1) + (x_2 + iy_2)) =$$

$$\alpha((x_1 + x_2) + i(y_1 + y_2)) = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -(y_1 + y_2) & -(x_1 + x_2) \end{bmatrix} =$$

$$\begin{bmatrix} x_1 & y_1 \\ -y_1 & -x_1 \end{bmatrix} + \begin{bmatrix} x_2 & y_2 \\ -y_2 & -x_2 \end{bmatrix} = \alpha(x_1 + iy_1) + \alpha(x_2 + iy_2),$$

$$\forall (x + iy) \in \mathbb{C}, k \in \mathbb{R}, \alpha(k(x + iy)) = \begin{bmatrix} kx & ky \\ -ky & -kx \end{bmatrix} = k \begin{bmatrix} x & y \\ -y & -x \end{bmatrix} = k \alpha(x + iy).$$

4. [20] The maps $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\beta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are defined by $\alpha(x, y, z) = (z, y, x)$ and $\beta(x, y, z) = (y + z, z + x, x + y)$.

(a) Write down the map $2\alpha + \beta$ and the values of $\alpha(1, 2, 3)$, $\beta(1, 2, 3)$ and $(\alpha - \beta)(1, 2, 3)$.

(b) Calculate the values of the compositions $(\beta\alpha)(1, 2, 3)$ and $(\alpha\beta)(1, 2, 3)$.

Solution

$$(a) \quad (2\alpha + \beta)(x, y, z) = 2\alpha(x, y, z) + \beta(x, y, z) =$$

$$2(z, y, x) + (y + z, z + x, x + y) = (y + 3z, x + 2y + z, 3x + y),$$

$$\alpha(1, 2, 3) = (3, 2, 1), \beta(1, 2, 3) = (2 + 3, 3 + 1, 1 + 2) = (5, 4, 3),$$

$$(\alpha - \beta)(1, 2, 3) = \alpha(1, 2, 3) - \beta(1, 2, 3) = (3, 2, 1) - (5, 4, 3) = (-2, -2, -2).$$

$$(b) \quad (\beta\alpha)(1, 2, 3) = \beta(3, 2, 1) = (2 + 1, 1 + 3, 3 + 2) = (3, 4, 5), \text{ and}$$

$$(\alpha\beta)(1, 2, 3) = \alpha(5, 4, 3) = (3, 4, 5).$$

Alternatively, $(\beta\alpha)(x, y, z) = \beta(z, y, x) = (y + x, x + z, z + y)$, so $(\beta\alpha)(1, 2, 3) = (3, 4, 5)$, and $(\alpha\beta)(x, y, z) = \alpha(y + z, z + x, x + y) = (x + y, z + x, y + z)$, so $(\alpha\beta)(1, 2, 3) = (3, 4, 5)$.

[Note that *in general* multiplication (or composition) of maps does not commute, although in this particular case it does!]

Maple Solution